NUMERICAL SOLUTION OF PROBLEMS PERTAINING TO A SUBMERGED JET IN POWER-LAW FLUIDS

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Power-law fluids are defined as the particular case of Stokes fluids for low Truesdell numbers. To describe motion in a submerged jet we employ boundary-layer type equations which are numerically solved on a Ural-2 computer.

§1. Definition of power-law fluids. In accordance with the classical concepts, stresses in a fluid are functions of the spatial velocity gradient. According to the principle of objectivity formulated by Noll [1], the stressed tensor in the rheological equation of state must be an isotropic function of the strain-rate tensor

$$p_{ij} = f(s_{ij}). \tag{1}$$

Fluids described by Eq. (1) are subdivided into two classes: Reiner-Rivlin fluids which exhibit a relaxation time, and Stokes fluids which exhibit no relaxation time [2]. For Stokes fluids Eq. (1) assumes the particular form

$$p_{ij} = f(s_{ij}, \mu_0, \theta_0),$$
$$\text{if } s_{ij} = 0, \text{ then } p_{ij} = -p\,\delta_{ij}. \tag{2}$$

Here μ_0 is the constant of the medium, and it is expressed in units of viscosity; θ_0 is a characteristic temperature (for example, the boiling point); p is the hydrostatic pressure; δ_{ij} is the Kronecker delta.

Following the usual rule for expansion in series in powers of the tensor and using the Cayley-Hamilton identity, instead of (2) we will have

$$p_{ij} = F_0\delta_{ij} + F_1 s_{ij} + F_2 s_{ik}s_{kj}, \tag{3}$$

where for an incompressible fluid $F_0 = -p$; F_1 and F_2 are functions of the strain-rate tensor invariance I_2,

n	0.5	0.7	1.0	2.0	3.0	4.0
$f'(0)$	0.18650	0.31100	0.45430	0.71166	0.83024	0.89794
γ	5.36187	2.44280	1.48305	1.00000	0.95455	0.95785

I_3 and the constants μ_0 and θ_0.

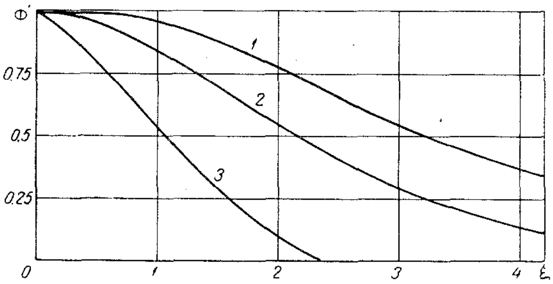
Since the complexes

$$E_0 = \frac{F_0}{p},\ E_1 = \frac{F_1}{\mu_0},\ E_2 = \frac{F_2 p}{\mu_0^2} \tag{4}$$

are dimensionless, for an incompressible fluid Eq. (3) assumes the following dimensionless form:

$$p_{ij} = pE_0\delta_{ij} + \left(1 + \frac{\mu_0 s_{ij}}{p}\,\frac{E_2}{E_1}\right)\mu_0 E_1 s_{ij}. \tag{5}$$

In formula (5) the dimensionless parameter Tr = $= \mu_0 s_{ij}/p$, known as the Truesdell number, is the criterion for the appearance of nonlinear effects.



Velocity profiles in jet for certain values of n: 1) n = 0.5; 2) 1; 3) 3.

In the following we will examine the case Tr ≪ 1, when the tensorial nonlinearity in (3) can be neglected, and the nonlinearity will be determined by the coefficient $F_1 = f(I_1, I_2, I_3)$. For simplicity we will study the case

$$F_1 = \mu_1 |2I_2|^{\frac{n-1}{2}}. \tag{6}$$

The validity of this relationship has been confirmed experimentally in [3]. The rheological equation (3) now assumes the following form:

$$p_{ij} = -p + \mu_1 |2I_2|^{\frac{n-1}{2}} s_{ij}. \tag{7}$$

In dimensionless form, the boundary-layer equations have the following form [4]:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left\{\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial u}{\partial y}\right\},$$
$$\frac{\partial p}{\partial y} = 0,\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{8}$$

§2. The problem of the submerged jet. The possibility of utilizing equations of the boundary-layer type to model motion in a submerged jet has been validated in [5]. Here $\partial p/\partial x \equiv 0$ and system (8) assumes the form

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left\{\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial u}{\partial y}\right\},$$
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{9}$$

Without carrying out the complete group analysis of system (9), let us write out the infinitesimal operators of the similarity group, with respect to which we have the invariance

$$X_1 = \frac{1}{2-n} u \frac{\partial}{\partial u} + \frac{n-1}{2-n} v \frac{\partial}{\partial v} + x \frac{\partial}{\partial x},$$

$$X_2 = \frac{n+1}{2-n} u \frac{\partial}{\partial u} + \frac{2n-1}{2-n} v \frac{\partial}{\partial v} - y \frac{\partial}{\partial y}.$$

Let us find the invariant-group solution determined from the combination of the operators X_1 and X_2. According to the general method [6], this solution must be determined from the condition

$$kX_1 + X_2 \equiv 0. \quad (10)$$

Then we will have

$$u = x^m J_1(\eta), \; v = x^{\frac{(2n-1)m-n}{n+1}} J_2(\eta), \; \eta = yx^{\frac{(2-n)m-1}{n+1}}. \quad (11)$$

We can demonstrate in the conventional manner [5] that the condition of conservation of momentum exists along the jet, i. e.,

$$\int_{-\infty}^{\infty} u^2 dy = 2 \int_0^{\infty} u^2 dy = 1,$$

which, with consideration of (11), assumes the form

$$2 \int_0^{\infty} J_1^2(\eta)\, d\eta = 1. \quad (12)$$

Substituting (11) into (9) with consideration of (12), with the usual boundary conditions [5] implicit, after introduction of the stream function

$$J_1 = f', \; J_2 = -\frac{1}{3n} f + \frac{2}{3n} \eta f' \quad (13)$$

we will have the boundary problem for the determination of f:

$$|f''|^{n-1} f''' + \frac{1}{3n}(ff'' + f'^2) = 0,$$

$$f(0) = f''(0) = 0, \; f'(\infty) = 0, \; 2\int_0^{\infty} f'^2 \, d\eta = 1. \quad (14)$$

Since $f'' \le 0$ in the submerged jet, it is convenient to carry out the following substitution of variables:

$$\varphi = -f, \quad \eta = \eta,$$

subsequent to which Eq. (14) assumes the form

$$\varphi''^{n-1} \varphi''' - \frac{1}{3n}(\varphi\varphi'' + \varphi'^2) = 0. \quad (15)$$

The integration of Eq. (15) with consideration of the boundary conditions in (14) leads to the following formula for the determination of the velocity profile:

$$\varphi' = (-1)^{\frac{n}{2n-1}} \left[c - \frac{(2n-1)\varphi^{\frac{n+1}{n}}}{\sqrt[n]{3}(n+1)} \right]^{\frac{n}{2n-1}}. \quad (16)$$

Here c is the magnitude of the velocity at the jet axis. Analysis of formula (16) shows that analytical solutions with physical significance do not exist for all n. When $n < 1$ the velocity profiles tend asymptotically toward zero as the argument approaches infinity, while for $n > 1$ the asymptotic property is disrupted. In this connection, certain of the results from [7] are cast in doubt.

Since Eq. (15) is invariant with respect to the similarity transform

$$\Phi = \gamma^{\frac{1-2n}{2-n}} \varphi, \quad \eta = \gamma\xi,$$

we can turn from the boundary problem (15) and (14) to the equivalent Cauchy problem

$$\Phi''^{n-1} \Phi''' - \frac{1}{3n}(\Phi\Phi'' + \Phi'^2) = 0,$$

$$\Phi(0) = \Phi''(0) = 0, \; \Phi'(0) = -1, \quad (17)$$

whose solution permits us to determine the unknown parameter γ according to the formula

$$\gamma = \frac{1}{\left[2\int_0^{\infty} \Phi'^2 \, d\xi\right]^{\frac{n-2}{3n}}}. \quad (18)$$

We note that near zero Eq. (17) exhibits a singularity, which is a serious inconvenience in numerical calculation. However, as $\xi \to 0$, $\Phi' \to -1$, $\Phi'' \to 0$, $\Phi \to 0$, Eq. (17) is equivalent to the following:

$$\Phi''^{n-1}\Phi''' - \frac{1}{3n} = 0. \quad (19)$$

Integrating (19), we have a representation for the function Φ near zero:

$$\Phi = -\xi \left(1 - \frac{n^2}{\sqrt[n]{3}(n+1)(2n+1)} \xi^{\frac{n+1}{n}}\right). \quad (20)$$

Now instead of (17) we have an original problem that is convenient for numerical realization:

$$\Phi''^{n-1}\Phi''' - \frac{1}{3n}(\Phi\Phi'' + \Phi'^2) = 0,$$

$$\text{when } \xi = \xi_0, \; \Phi = \Phi_0, \; \Phi' = \Phi_0', \; \Phi'' = \Phi_0''. \quad (21)$$

The quantities Φ_0, Φ_0' and Φ_0'' are determined in this case from formula (20).

System (21) was solved according to the Runge-Kutta formula with automatic selection of the pitch for a specified calculation accuracy on the order of 10^{-6}; the integral in formula (18) was calculated in accordance with the Simpson formula. All of the calculations were carried out on a Ural-2 computer. The quantity ξ_0 was determined experimentally. We know that with n equal to unity Eq. (14) has an exact solution, and the unknown value of the velocity at the jet axis $f'(0)$ is equal to 0.454 [5]. Assuming the quantity ξ_0 to be equal to 10^{-3}, solving (21) numerically, and calculating γ according to (18), we find that $f'(0)$ equals 0.45430. We regard this agreement as satisfactory and assume in the following that ξ_0 is equal to 10^{-3}. For purposes of comparison we present the values of the velocity at the jet axis $f'(0)$ for various values of n:

The figure shows the profiles of the velocity $\Phi'(\xi)$ for several n. Analysis of the cited results shows that with an increase in n there is an increase in the velocity at the jet axis, while for n smaller than unity, the profiles are fuller than when n is larger than unity.

NOTATION

x and y are the longitudinal and transverse coordinate; u and v are the longitudinal and transverse velocities in the boundary layer; p_{ij} is the tensor; s_{ij} is the strain-rate tensor; p is the hydrostatic pressure; I_1, I_2, and I_3 are the invariants of the strain-rate tensor.

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